GROUP-THEORETIC ORIGIN OF CERTAIN GENERATING FUNCTIONS

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1. Introduction. A linear ordinary differential equation containing a parameter n may be written in the form

$$L(x, d/(dx), n)v=0 . \tag{1.1}$$

Substituting $A=y\partial/(\partial y)$ for n, supposing the left member a polynomial in n, we construct the partial differential operator $L=L(x, d/(dx), A)$ on functions of two independent variables. This operator is independent of n and is commutative with A. A solution of the simultaneous equation $Lu=0$, $Au=nu$, where n is a constant, has the form $u=v_n(x)y^n$, where $v=v_n(x)$ is a solution of (1.1). Conversely, if $v=v_n(x)$ is a solution of (1.1), then $u=v_n(x)y^n$ is a solution of the equations $Lu=0$, $Au=nu$.

Now suppose that, independently of the preceding considerations, *we have obtained an explicit solution* $u=g(x, y)$ *of* $Lu=0$, *and that* from the properties of this function we know that it has an expansion in powers of y of the form

$$g(x, y)=\sum_n g_n(x)y^n , \tag{1.2}$$

where n is not necessarily an integer. If termwise operation with L on this series is permissible, then L annuls each term of the series, and $v=g_n(x)$ is a solution of (1.1). Thus $g(x, y)$ is a generating function for certain solutions of (1.1). The main problem is to find $g(x, y)$; its expansion is a detail of calculation.

It is difficult, in general, to find an explicit solution of $Lu=0$, other than an artificial superposition of the functions $v_n(x)y^n$, for which the generating function reduces to a tautology. However, if the equation admits a group of transformations besides $x'=x$, $y'=ty$ ($t\neq 0$), it is possible, in many cases, to find a solution which leads to a significant generating function of the form (1.2). In this paper it will be shown in detail how generating functions for the hypergeometric functions $F(-n, \beta ; \gamma ; x)$ may be obtained by this method. The Kummer functions ${}_1F_1(-n ; \gamma ; x)$ and ${}_1F_1(\alpha ; n+1, x)$, the Bessel functions $J_n(x)$ and the Hermite functions $H_n(x)$ admit similar treatment. The point to be emphasized is that the generating functions so obtained owe *their existence to the fact that the partial differential* equation derived from the ordinary differential equation in the manner described above is invariant with respect to a nontrivial continuous group of transformations.

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