WEAK LOCALLY MULTIPLICATIVELY-CONVEX ALGEBRAS[1]

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Let E be an algebra over the reals or complex numbers, E' a total subspace of the algebraic dual E^* of vector space E. We first discuss the following natural questions: When is the weak topology $\sigma(E, E')$ defined on E by E' locally m-convex? When is multiplication continuous for $\sigma(E, E')$, that is, when is $\sigma(E, E')$ compatible with the algebraic structure of E? We then apply our results to certain weak topologies on the algebra of polynomials in one indeterminant without constant term.

1. Weak topologies.

Let K be either the reals or complex numbers, E a K-algebra. A topology $\mathscr{T}$ on E is *locally multiplicatively-convex* (which we abbreviate henceforth to "locally m-convex") if it is a locally convex topology and if there exists a fundamental system of idempotent neighborhoods of zero (a subset A of E is idempotent if $A^2 \subseteq A$). Multiplication is then clearly continuous at (0, 0) and hence everywhere, so $\mathscr{T}$ is compatible with the algebraic structure of E. If A is idempotent, so is its convex envelope, its equilibrated envelope (a subset V of E is called equilibrated if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$), and its closure for any topology on E compatible with the algebraic structure of E. Hence if $\mathscr{T}$ is locally m-convex, zero has a fundamental system of convex, equilibrated, idempotent, closed neighborhoods. (For proofs of these and other elementary facts about locally m-convex algebras, see §§1–3 of [8] or [1].) Henceforth, E' is a total subspace of the algebraic dual of E.

LEMMA 1. *Let W be a weak, equilibrated neighborhood of zero (that is, for the topology $\sigma(E, E')$), J a subspace of E, and $g \in E'$ such that $J \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$. Then J, JE, and EJ are contained in the kernel of g.*

Proof. Let $x \in J$, $y \in E$. As W is equilibrated and absorbing, let $\lambda > 0$ be such that $\lambda y \in W$. For all positive integers m, $\lambda^{-1}mx \in J$, and therefore $mxy = (\lambda^{-1}mx)(\lambda y) \in JW \subseteq W^2 \subseteq \{g\}^0$; hence $|g(mxy)| \leq 1$ for all positive integers m, and therefore $g(xy) = 0$. Hence JE is contained in

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