THE MAPPINGS OF THE POSITIVE INTEGERS INTO THEMSELVES WHICH PRESERVE DIVISION

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1. Introduction, First Theorem. Let L denote the lattice of the integers $0, 1, 2, \cdots$ partially ordered by division. We study here mappings

$$\phi: \quad \phi_0, \phi_1, \phi_2, \cdots, \phi_n = \phi(n), \cdots$$

of L into itself which preserve division; that is,

(i) *If n divides m, then ϕ_n divides ϕ_m.*

Since ϕ_1 divides every ϕ_n and every ϕ_n divides ϕ_0, we lose little generality by assuming

(ii) $\phi_0 = 0, \quad \phi_1 = 1.$

Any mapping with properties (i) and (ii) will be called a divisibility sequence on L.

A mapping ϕ is said to be of "*positive character*" if

(iii) $\phi_n > 0$ for $n > 0$.

A divisibility sequence of positive character will be called a *normal sequence* or *normal mapping* of L.

In many instances, we are interested in the occurrence of multiples of some assigned modulus m among the terms of a normal sequence ϕ. If $\phi_r \equiv 0 \pmod{m}$ for some $r > 0$, we call m a *divisor* of ϕ and r a "*place of apparition*" of m in ϕ. If in addition $\phi_s \not\equiv 0 \pmod{m}$ for every proper divisor s of r, r is called a "*rank of apparition*" of m in ϕ. If m is not a divisor of ϕ, we assign to it the rank of apparition zero, which is consistent with the definitions.

It follows that every modulus m has at least one rank of apparition in ϕ. If each modulus has exactly one rank of apparition, we say that ϕ "*admits a rank function*". Indeed if the rank of m in ϕ is denoted by $\rho(m)$ then ρ is a divisibility sequence. Furthermore

(iv) $\phi_n \equiv 0 \pmod{m}$ *if and only if* $n \equiv 0 \pmod{\rho_m}$.

Under this condition, multiples of any integer m if they appear at all in ϕ are regularly spaced as in the identity mapping $i(n) = n$.

Normal sequences are of common occurrence in number theory; the totient function and its various generalizations [3, chap. 5] is a

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