ON THE NUMERICAL SOLUTION OF POISSON'S EQUATION OVER A RECTANGLE

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Introduction. We consider the equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=f(x, y)$$

over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, with given boundary values for z. Following the usual procedure (see for example Hyman [1]) we approximate the solution by solving a set of mn simultaneous equations, arising from the corresponding difference equation. If we write

$$a=(n+1)\Delta x, \quad b=(m+1)\Delta y, \quad \rho=\Delta y/\Delta x, \quad a_{i,j}=-f(j\Delta x, i\Delta y)\Delta y^2$$

and $z_{i,j}=z(j\Delta x, i\Delta y)$, the mn equations are of the form

$$(1) \qquad 2(1+\rho^2)z_{i,j}=\rho^2(z_{i,j+1}+z_{i,j-1})+z_{i+1,j}+z_{i-1,j}+a_{i,j}\,, \qquad i=1,\cdots,m, \quad j=1,\cdots,n\,.$$

A solution of this set of equations is given by Hyman [1]. In the case where the boundary values are zero, the solution takes the form $Z=C\omega D$ [1, p. 340] where C and D are matrices which depend on n and m and may be written down without any calculations, and ω is a matrix depending on m, n, ρ and the values of $f(x, y)$ at the lattice points. The matrix ω requires somewhat elaborate calculations. To obtain the solution with given boundary values, he adds to the matrix $C\omega D$ the value of u as a matrix obtained from the solution of the equation $\Delta^2 u/\Delta x^2+\Delta^2 u/\Delta y^2=0$ with the given boundary values. He obtains for u the matrix value $U=C\phi$ [1, p. 329], where C is the matrix mentioned above and ϕ is a matrix depending on n, m, ρ and the boundary values and requires to be recalculated for every set of boundary values.

In this paper the solutions of equations (1) are obtained, column by column, in the form $Z_j=\sum_k M_{j,k}B_k$, where the $M_{j,k}$ are matrices depending on m, n, and ρ and which require somewhat elaborate calculations, and the B_k are vectors depending on m, n, ρ, the values of $f(x, y)$ at the lattice points and the boundary values and can be written down without calculation. We may regard this solution as giving an explicit formula for the values of z at the lattice points.

The principal work in the calculation of Z_j is the calculation of the matrices $M_{j,k}$. It will be shown that it is sufficient to calculate a

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