ON THE TOWER THEOREM FOR FINITE GROUPS

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Wielandt [2] has given a very ingenious proof of the fact that the tower of automorphisms of a finite group without center ends after a finite number of steps. Using his work as a model a proof of a similar tower theorem for Lie algebras was given in [1]. This depends on the following three facts:

(a) If A (with no center) is a member of the tower of derivation algebras of a Lie algebra L then the centralizer of L in A is (0).

(b) If L is a subinvariant Lie algebra of A and if the centralizer of L in A is (0) then the centralizer of L^{ω} in A is contained in A.

(c) If L is subinvariant in A then L^{ω} is normal in A.

In view of the much sharper estimate obtained in the theorem on Lie algebras it seemed to be of interest to attempt to improve on the results of Wielandt using the method of [1]. The group theory analogue of (a) is to be found in Wielandt's work. I shall prove here the analogue to (b) and then show by a counter-example that the method is not applicable to get the tower theorem even for solvable groups since the analogue to (c) does not hold for groups even under the additional hypothesis of (b).

THEOREM. *If G is a subinvariant subgroup of the finite group A and if the centralizer of G in A is the identity, then the centralizer of G^{ω} in A is contained in G^{ω}. It follows that if N is normal in G such that G/N is nilpotent then $N \supset G^{\omega}$ and the centralizer of N is contained in N.*

Here

$$G^{\omega} = \bigcap_{k=1}^{\infty} G^k$$

where $G^k = [G^{k-1}, G]$ is the subgroup generated by commutators of the form $[h, g] = hgh^{-1}g^{-1}$, $h \in G^{k-1}$, $g \in G$.

The proof of the Theorem depends on two lemmas.

LEMMA 1. *If G is a finite group then $G = G^{\omega}H$ where H is a nilpotent subgroup of G.*

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