A COMPARISON THEOREM FOR EIGENVALUES OF NORMAL MATRICES

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The following interesting theorem was recently obtained by H. Wielandt (Oral communication, see also J. Todd [3]):

Let M, N be two normal matrices of order n, and let r denote the rank of $M-N$. Let D be an arbitrary closed circular disk in the complex plane, If D contains exactly p eigenvalues of M, and exactly q eigenvalues of N, then $|p-q|\leq r$.

It is then natural to raise the following question: Without considering the rank of $M-N$, is it possible to compare the eigenvalues of M and N in a manner similar to that of Wielandt's theorem? The purpose of this Note is to present such a rank-free comparison theorem which includes Wielandt's theorem stated above.

THEOREM. *Let M, N be two normal matrices*[1] *of order n and let r be an integer such that $0\leq r<n$. Let $\varepsilon\geq 0$ be such that ε^2 is not less than the* $(r+1)$th *eigenvalue of $(M-N)^*(M-N)$, when the eigenvalues of $(M-N)^*(M-N)$ are arranged in descending order.*[2] *If a closed circular disk*

$$|z-z_0|\leq\rho$$

contains p eigenvalues of M, then the concentric disk

$$|z-z_0|\leq\rho+\varepsilon$$

contains at least $p-r$ eigenvalues of N.

While Wielandt's proof of his theorem uses geometric arguments involving convexity, the proof of our theorem will be based on an inequality (Lemma below). This difference in methods explains why our result is of more quantitative character than Wielandt's theorem.

LEMMA. *Let A, B be any two matrices*[3] *of order n. If $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ are the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$ respectively, each arranged in descending order*

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[1] The elements of all matrices considered here are real or complex numbers.

[2] As usual, the adjoint of a matrix A is denoted by A^*.

[3] Here A, B need not be normal.