PSEUDO-DISCRIMINANT AND DICKSON INVARIANT

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1. Let E be a vector space of finite dimension over a field K. To a bilinear symmetric form $f(x, y)$ defined over $E \times E$ is attached classically the notion of *discriminant*: it is an element of K which is not entirely defined by f; however, it is entirely determined when in addition a basis of E is chosen, and when the basis is changed, the discriminant is multiplied by a *square* in K. More precisely, let u be a linear mapping of E into E, and let $f_1(x, y) = f(u(x), u(y))$ the form "transformed" by u; if $\Delta(f)$, $\Delta(f_1)$ are the discriminants of f and f_1 with respect to the *same* basis of E, and $D(u)$ the determinant of u with respect to that basis, then one has the classical relation

$$\Delta(f_1) = (D(u))^2 \Delta(f) . \tag{1}$$

When K has characteristic $\neq 2$, the preceding results may be expressed in terms of the "quadratic form" $f(x, x)$ associated to $f(x, y)$. However, when K has characteristic 2, the one-to-one association between bilinear symmetric forms and quadratic forms no longer subsists. More precisely, to a given *alternate* symmetric form $f(x, y)$ (that is, $f(x, x) = 0$ for all $x \in E$) is associated a whole family of quadratic forms $Q(x)$, satisfying the fundamental identity

$$Q(x+y) = Q(x) + Q(y) + f(x, y) \tag{2}$$

and to all these Q is associated the same discriminant of f (with respect to a given basis).

Now C. Arf [1] has introduced an element $\Delta(Q)$ attached to Q and to a given *symplectic basis* of E (with respect to the form f) which we shall call the *pseudo-discriminant* of Q. He proved moreover that under a change of symplectic basis, $\Delta(Q)$ is transformed in the following way: let $\wp$ be the homomorphism $\xi \rightarrow \xi + \xi^2$ of the additive group K into itself; then the pseudo-discriminants of Q with respect to two different symplectic bases have a *difference* which has the form $\wp(\lambda)$. Arf's proof is rather lengthy and proceeds by induction on n. We propose to show how the pseudo-discriminant is related to the *Clifford algebra* of Q in a way which parallels the well-known relation between the discriminant of f and the Clifford algebra of f over a field of characteristic $\neq 2$. At the same time, this will clear up the origin of a curiously isolated result obtained by L. E. Dickson for the orthogonal

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