SPECIALIZATIONS OVER DIFFERENCE FIELDS

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Introduction. We consider a system S of algeraic difference equations with coefficients in a difference field $\mathscr{F}$ and involving also parameters λ_i. Well-known results concerning systems of algebraic equations and systems of algebraic differential equations would lead one to expect that, if S has solutions in some extension of the difference field formed by adjoining the parameters λ_i to $\mathscr{F}$, then the system resulting from S by assigning special values to the λ_i has solutions, provided only that the special values are chosen so as not to annul a certain difference polynomial. But the examples in [5, p. 510] show that this is not so.

The difficulty in these examples arises from the fact that a difference field $\mathscr{F}$ may have incompatible extensions, that is to say, extensions which cannot both be embedded isomorphically in any one of its extensions. In particular, it may happen that one can express in terms of a solution of the system S an element α, independent of the λ_i. Then α will be contained in the difference field formed by adjoining to $\mathscr{F}$ a solution of any system (possessing solutions) which arises by specializing the parameters of S. It will then not be possible to find solutions if one specializes the λ_i in such a way that the extension of $\mathscr{F}$ formed by adjoining the specialized values is incompatible with that formed by adjoining α.

The principal result of this paper is that one can restore the expected result concerning the specialization of parameters of S by imposing a suitable condition of compatibility. If the system S has solutions, then, in order to assure that the system obtained from S by specializing the parameters has solutions, it suffices to choose the specializations from an extension of $\mathscr{F}$ compatible with a certain extension $\mathscr{G}$ of $\mathscr{F}$ and not annulling a certain difference polynomial. In particular, if $\mathscr{F}$ is algebraically closed it has no incompatible extensions so that it suffices to choose specializations of the parameters not annulling a certain difference polynomial. Hence, in this case, one has the same freedom of specialization as with systems of algebraic equations. Even in the general case, there is considerable freedom as the compatibility condition will evidently be satisfied if the specialized values are chosen from $\mathscr{G}$ itself or any extension of $\mathscr{G}$. We turn now to a formal discussion of this theorem.

We consider a difference field $\mathscr{F}$ and extensions $\mathscr{G}$ and $\mathscr{H}$ of

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