SOME ERGODIC THEOREMS INVOLVING TWO OPERATORS

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1. Introduction. The object of the present note is to indicate how the ergodic theorem of W. Hurewicz [3] and E. Hopf [2] can be extended to theorems involving two operators. While for a finite measure space, the Hopf theorem for two operators is readily seen to be the consequence of the theorem for one operator and the Birkhoff ergodic theorem, in the general case the theorem for two operators is established via the extended form of the Hurewicz theorem. An application is made to the theory of Markov chains in § 4.

Let (S, Ω, μ) be a fixed measure space which is assumed to be σ-finite unless otherwise stated. Capital letters are reserved for elements of Ω. For a measure ξ and for point functions we write $f(x)=g(x)[\xi]$ for equality almost everywhere $[\xi]$.

We consider two one-to-one transformations of S onto itself, t and u, each of which is measurable in the sense that for $v=t$ and $v=u$, $M \in \Omega$ implies $vM \in \Omega$ and $v^{-1}M \in \Omega$, and if $\mu(M)=0$ then $\mu(v^{-1}M)=0$. We suppose throughout that neither t nor u has wandering sets of positive measure, that is,

(1) *For* $v=t$ *and* $v=u$, *if* $A \cap v^k A=0$, $k=1, 2, \cdots$, *then* $\mu(A)=0$.

2. The Hurewicz theorem. For any finite valued countably additive set function φ defined on Ω and absolutely continuous with respect to μ, form the set functions

$$\varphi_n(X)=\sum_{k=0}^{n}\varphi(t^k X), \qquad n=0, 1, \cdots, \tag{2}$$

and

$$\nu_n(X)=\sum_{k=0}^{n}\mu(t^k X), \qquad n=0, 1, \cdots. \tag{3}$$

Then φ_n and ν_n are countably additive set functions and φ_n is absolutely continuous with respect to ν_n so admits the representation

$$\varphi_n(X)=\int_X g_n(x)\mu_n(dx), \qquad n=0, 1, \cdots. \tag{4}$$

The Hurewicz theorem then asserts that $g_n(x)$ has a limit at all points except for a nullset with respect to t, that is for all points except a t-invariant set of μ measure zero.

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