ABSTRACT RIEMANN SUMS

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1. Introduction. A theorem of B. Jessen [5] asserts that for $f(x)$ of period one and Lebesgue integrable on $[0, 1]$

$$(1) \qquad \lim_{n\to\infty} 2^{-n}\sum_{k=0}^{2^n-1} f(x+k2^{-n})=\int_0^1 f(t)dt \text{ almost everywhere}.$$

We show that the theorem of Jessen is a special case of a theorem analogous to the Birkhoff ergodic theorem [1] but dealing with sums of the form

$$(2) \qquad 2^{-n}\sum_{k=0}^{2^n-1} f(T^{k/2^n}x).$$

In this form T is an operator on a σ-finite measure space such that $T^{1/2^n}$ exists as a one-to-one point transformation which is measure preserving for $n=0, 1, \cdots$, and $f(x)$ is integrable with $f(x)=f(Tx)$. We also obtain in § 3 the analogues for abstract Riemann sums of the ergodic theorems of Hurewicz [4] and of Hopf [3].

We might remark that there is no use, due to the examples of Marcinkiewicz and Zygmund [6] and Ursell [8], in considering sums of the form

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^{k/n}x)$$

without further hypothesis on $f(x)$. However we may replace 2^n throughout by $m_1m_2\cdots m_n$ with m_j integral and $m_j\geq 2$ without altering any argument.

In § 4 necessary and sufficient conditions are obtained on a transformation T in order that the sums (2) have a limit as $n\to\infty$ for almost all x. These conditions are analogous to those of Ryll-Nardzewski [7] in the ergodic case. We use the necessary conditions to establish an analogue of a form of the Hurewicz ergodic theorem for two operators [2].

2. Notation. Let (S, Ω, μ) be a fixed σ-finite measure space. We consider throughout point transformations T which have measurable square roots of all orders, that is,

(3.1) *There exist one-to-one point transformations T_n so that*

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