THE USE OF FORMS IN VARIATIONAL CALCULATIONS

Louis Auslander

Introduction. The purpose of this paper is to present a method of calculating the first and second variation which is suitable for spaces which have a Euclidean connection. I then use this method to calculate the first and second variations along a geodesic in a Finsler space in terms of differential invariants of the Finsler metric. In the special case of Riemannian geometry, this calculation has been carried out by Schoenberg in [**4**].

Indications as to how this calculation should be made are originally due to E. Cartan [**1**]. I wish to thank Prof. S. S. Chern for the privilege of seeing his calculations on this matter for Riemann spaces.

1. Algebraic Preliminaries. Let $I=[0, 1]$ and $0\leq\xi_1, \xi_2\leq 1$. Let M^n be an n-dimensional C^∞ manifold. Assume we have a one parameter family of mappings of I into M^n which we will denote by $f(\xi_1, \xi_2)$, where ξ_2 is taken as the parameter along I and ξ_1 parametrizes the family of mappings. Then we may define a mapping η: $I\times I\rightarrow M^n$ by the equation

$$\eta(\xi_1, \xi_2)=f(\xi_1, \xi_2).$$

We require that η shall also be a C^∞ mapping.

Let η_* denote the mapping induced by η on the tangent space to $I\times I$ into the tangent space to M^n. Let η^* denote the dual mapping induced on the cotangent spaces. Then we define two vector fields X_1 and X_2 over $\eta(I\times I)$ by

$$X_2=\eta_*(\partial/\partial\xi_2) \quad \text{and} \quad X_1=\eta_*(\partial/\partial\xi_1).$$

Then if w is any form in M^n we may write

$$\eta^*(w)=w_\delta d\xi_1+w_d d\xi_2,$$

where w_δ and w_d are defined by the equation.

Lemma 1.1. *If $\langle X, w\rangle$ denotes the value that X takes on the covector w at each point, then*

$$w_\delta=\langle X_1, w\rangle$$

and

$$w_d=\langle X_2, w\rangle.$$

Received February 12, 1954 and in revised form June 2, 1954.