ON A THEOREM OF S. BERNSTEIN

N. C. ANKENY AND T. J. RIVLIN

1. Introduction and proof of the main theorem. A result of S. Bernstein [4] is the following.

THEOREM A. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *then*

$$[\max|p(z)|,\ |z|=R>1]\leq R^n, \tag{1}$$

with equality only for $p(z)=\lambda z^n$, *where* $|\lambda|=1$.

We propose to show here that if we restrict ourselves to polynomials of degree n having no zero within the unit circle the right hand member of (1) can be made smaller. In particular we have the following result.

THEOREM 1. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *and* $p(z)$ *has no zero within the unit circle, then*

$$[\max|p(z)|,\ |z|=R>1]\leq\frac{1+R^n}{2},$$

with equality only for $p(z)=(\lambda+\mu z^n)/2$, *where* $|\lambda|=|\mu|=1$.

In order to prove Theorem 1 we use a conjecture of Erdös first proved by Lax [2] (See also [1]).

THEOREM B. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *and* $p(z)$ *has no zero within the unit circle, then*

$$[\max|p'(z)|,\ |z|=1]\leq\frac{n}{2}.$$

Turning now to Theorem 1, let us assume that $p(z)$ does not have the form $(\lambda+\mu z^n)/2$. In view of Theorem B

$$|p'(e^{i\varphi})|\leq\frac{n}{2},\qquad 0\leq\varphi<2\pi, \tag{2}$$

from which we may deduce that

Received August 4, 1954. This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.