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THE NUMERICAL RANGE OF AN OPERATOR

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Let A be a continuous linear operator on a complex Hilbert space X with inner product $<, >$ and associated norm $\|\ \|$. Let $W(A) = \{\langle Ax, x\rangle \mid \|x\| = 1\}$ be the numerical range of A and for each complex number z let $M_z = \{x \mid \langle Ax, x\rangle = z\|x\|^2\}$. Let $\curlyvee M_z$ be the linear span of M_z and $M_z \oplus M_z = \{x + y \mid x \in M_z$ and $y \in M_z\}$. An element z of $W(A)$ is characterized in terms of the set M_z as follows:

THEOREM 1. If $z \in W(A)$, then $\curlyvee M_z = M_z \oplus M_z$ and

(i) z is an extreme point of $W(A)$ if and only if M_z is linear;

(ii) if z is a nonextreme boundary point of $W(A)$, then $\curlyvee M_z$ is a closed linear subspace of X and $\curlyvee M_z = \cup \{M_w \mid w \in L\}$, where L is the line of support of $W(A)$, passing through z. In this case $\curlyvee M_z = X$ if and only if $W(A) \subset L$.

(iii) if $W(A)$ is a convex body, then x is an interior point of $W(A)$ if and only if $\curlyvee M_z = X$.

It is well-known that $W(A)$ is a convex subset of the complex plane. Thus if $z \in W(A)$, either z is an *extreme point* (not in the interior of any line segment with endpoints in $W(A)$), a nonextreme boundary point, or an interior point (with respect to the usual plane topology) of $W(A)$. Thus Theorem 1 characterizes every point of $W(A)$.

The following additional notation and terminology are used. If $K \subset X$, then $K^\perp$ denotes the orthogonal complement of K. An operator A is *normal* if and only if $AA^* = A^*A$ and *hyponormal* only if $AA^* \ll A^*A$. A line L is a *line of support* for $W(A)$ if and only if $W(A)$ lies in one of the closed half-planes determined by L and $L \cap \overline{W(A)} \neq \emptyset$.

In the last section of the paper consideration is given to $\bigcap$ {maximal linear subspaces of M_z}. One result is that if A is hyponormal and z a boundary point of $W(A)$, then $\bigcap$ {maximal linear subspaces of M_z} $= \{x \mid Ax = zx$ and $A^*x = z^*x\}$. This generalizes Stampfli's result in [3]: if A is hyponormal and z is an extreme point of $W(A)$, then z is an eigenvalue of A. In [2] MacCluer proved this theorem for A normal.

2. A proof of Theorem 1.

Lemmas 1 and 2 provide the core of the proof of Theorem 1.

LEMMA 1. *Let z be in the interior of a line segment with endpoints a and b in $W(A)$, $x \in M_a$, $y \in M_b$, $\|x\| = \|y\| = 1$. There exist*