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A MEAN VALUE THEOREM FOR BINARY DIGETS

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This paper continues the investigation of the dyadically additive function α defined by $\alpha(n)=$ the number of 1's in the binary expansion of n.

Previously, Bellman and Shapiro (cf. "On a problem in additive number theory." Annals of Mathematics, 49 (1948) 333–340) showed that $\Sigma_{k=1}^{x}\alpha(k)\sim x\log x/2\log 2$. They then considered the iterates of α defined by $\alpha_q=\alpha_{q-1}\circ\alpha$ and observed that $A_r(x)=\Sigma_{k=1}^{x}\alpha_r(k)$ is not asymptotic to any elementary function for $r\geqq 2$.

In this paper the function $A_2(x)$ will be examined more closely. Defining $c(x)$ by $A_2(x)=c(x)x\log\log x/2\log 2$, we will prove the following theorems.

THEOREM 1. *As x ranges over the positive integers, $c(x)$ ranges densely over $[1/2,3/2]$. Furthermore, given any $c\in[1/2,3/2]$, there is an explicit way to construct a sequence of integers x for which $c(x)\rightarrow c$ as $x\rightarrow\infty$.*

THEOREM 2.

$$1/2+O(\log\log\log x/\log\log x)\leqq c(x)\leqq 3/2+O(\log\log\log x/\log\log x). \tag{1.1}$$

THEOREM 3.

$$\liminf c(x)=1/2,\qquad \limsup c(x)=3/2. \tag{1.2}$$

Note. Theorem 3 is an immediate consequence of Theorems 1 and 2.

2. The proof of Theorem 1 is obtained by considering a special set of integers.

Let $\mathcal{M}=\{x: x=2^M-1,\ M$ even, $M/2=\Sigma_{i=1}^{r}2^{a_i}-1,\ a_1>a_2>\cdots>a_r$ positive integers, and $a_r/a_1\geqq 1/2+\log\log\log x/\log\log x\}$.

LEMMA 1. *If $x\in\mathcal{M}$, then*