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TAUTNESS FOR ALEXANDER–SPANIER COHOMOLOGY

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The purpose of this note is to give a straightforward unified proof of the tautness of Alexander–Spanier cohomology in the cases where it is known to be valid and to give a necessary condition that every closed (arbitrary) subspace be taut with respect to zero dimensional cohomology.

Let F denote a contravariant functor from the category of topological spaces to the category of abelian groups. A subspace A of a topological space X is said to be *taut with respect to* F if the canonical map $\varinjlim\{F(U)\} \to F(A)$ is an isomorphism (the direct limit is taken over the family of all neighborhoods of A in X, the family being directed downward by inclusion). The subspace A is *taut* in X if it is taut with respect to the Alexander–Spanier cohomology theory $\bar{H}$ for every dimension and every coefficient group (for notation and terminology dealing with $\bar{H}$ see [6]).

This concept of tautness has proved to be important. In [6] and [7] it is shown that a closed subspace of a paracompact Hausdorff space is taut, and this is used to deduce a strong excision property for $\bar{H}$. This tautness property is also used in [6] to derive the continuity property for $\bar{H}$. In [4] it is shown that an arbitrary subspace of a metric space is taut with respect to Čech cohomology, and this is used to obtain a general duality in spheres. Since the Čech cohomology is isomorphic to $\bar{H}$ [3], every subspace of a metric space is taut. In [2] it is shown that every neighborhood retract of X is taut in X, and this is used to prove a generalized homotopy property for compact spaces. In [1] tautness is considered for sheaf cohomology and used in proving the Vietoris–Begle mapping theorem.

We shall prove a simple lemma which gives a sufficient condition for tautness. This sufficient condition is enough to establish tautness in all the various cases where it is known.

Let $\mathcal{U}$ be a collection of subsets of X and A a subset of X. The *star of* A *with respect to* $\mathcal{U}$, denoted by $\mathrm{st}(A, \mathcal{U})$, is defined to be the union of those elements of $\mathcal{U}$ whose intersection with A is nonempty. An *open covering of* A *in* X is a collection $\mathcal{U}$ of open sets of X such that $A \subset \mathrm{st}(A, \mathcal{U})$.

The following seems to be the main fact underlying tautness (see [2] and [6]).