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ON THE UNITARY INVARIANCE OF THE NUMERICAL RADIUS

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A characterization is obtained of scalar multiples of unitary matrices in terms of the unitary invariance of a generalized numerical radius. The method of proof involves some rather delicate combinatorial considerations.

1. Introduction. Let n and m be positive integers, $1 \leqq m \leqq n$, and denote by $M_{n,m}(\mathbf{C})$ ($M_n(\mathbf{C})$) the vector space of all n-by-m (n-square) complex matrices. For a matrix $A \in M_n(\mathbf{C})$, define the mth *decomposable numerical range of* A to be the set

$$W_m^\wedge(A) = \{\det(X^*AX) \mid X \in M_{n,m}(\mathbf{C}), \det(X^*X) = 1\} \tag{1}$$

in the complex plane (the reason for this choice of terminology will become apparent in the next section). It is not difficult to verify that $W_m^\wedge(A)$ is compact, so it makes sense to define the mth *decomposable numerical radius of* A by

$$r_m^\wedge(A) = \max_{z \in W_m^\wedge(A)} |z|. \tag{2}$$

When $m = 1$, $W_1^\wedge(A)$ is simply the classical *numerical range*

$$W(A) = \{(Ax, x) \mid x \in \mathbf{C}^n, \|x\| = 1\} \tag{3}$$

(here $(\cdot,\cdot)$ denotes the standard inner product in the space $\mathbf{C}^n$ of complex n-tuples), and $r_1^\wedge(A)$ is the classical *numerical radius*

$$r(A) = \max_{z \in W(A)} |z|. \tag{4}$$

The numerical radius $r(A)$ satisfies the interesting *power inequality*

$$r(A^k) \leqq r(A)^k, \quad k = 1, 2, 3, \cdots \tag{5}$$

[2, §176]. In general, the number $r_m^\wedge(A)$ is an important function of the matrix A. For example, it is a bound for the moduli of all products of m eigenvalues of A. This is an immediate consequence of Proposition 1. Another easy consequence (Corollary 2) of Proposition 1 is that if A