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RATIONAL HOMOTOPY AND UNIQUE FACTORIZATION

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Decompositions of various kinds of mathematical objects as products or coproducts are considered, and the uniqueness of these decompositions is discussed. For instance, the following topological result is proved. Let Δ be the set of formal, simply-connected, rational homotopy types having finitely generated rational homotopy. Of course, Δ is a commutative semigroup with respect to the usual product space construction. Then Δ is a *free* commutative semigroup.

This is a statement about "unique factorization" in Δ, and it follows from our main result (Theorem 4) concerning the unique factorization of certain differential graded Q-algebras called formal minimal algebras.

These results are reminiscent of the Krull–Schmidt Theorem (cf. [7], page 58), which is a unique factorization result for suitable classes of "M-groups". However, the Krull–Schmidt Theorem discusses (categorical) products, while our algebraic results are concerned with tensor product decompositions where the tensor product is the (categorical) coproduct.

In contrast to Corollary 8, negative results in finer topological contexts are obtained from the interesting noncancellation example of Hilton and Roitberg [5]: E is a compact, simply-connected manifold such that $S^3 \times \mathrm{Sp}(2)$ and $S^3 \times E$ are diffeomorphic; however, Sp(2) and E have distinct homotopy types. Thus, differentiable manifolds, topological spaces, and homotopy types each fail to satisfy the unique factorization property (cf. definition just before Theorem 2).

Several of the proofs in this paper require a discussion of rational homotopy theory, in the form of Sullivan's theory of "minimal models". A demonstration of the scope and depth of this beautiful theory may be found in [3] and [9], while [4] is a clear, self-contained introduction to this view of rational homotopy theory.

2. Splittings of minimal algebras. For the purpose of this paper, the term "minimal algebra" is restricted to the following (somewhat limited) definition (cf. [4]): A minimal algebra M consists of a simply-connected, free, associative, graded-commutative Q-algebra M,