CHAPTER II

6. Preliminary Lemmas of Lie Type

Hypothesis 6.1.

(i) *p is a prime, $\mathfrak{P}$ is a normal S_p-subgroup of $\mathfrak{P}\mathfrak{U}$, and $\mathfrak{U}$ is a non identity cyclic p'-group.*

(ii) $C_{\mathfrak{U}}(\mathfrak{P}) = 1$.

(iii) *$\mathfrak{P}'$ is elementary abelian and $\mathfrak{P}' \subseteq \boldsymbol{Z}(\mathfrak{P})$.*

(iv) *$|\mathfrak{P}\mathfrak{U}|$ is odd.*

Let $\mathfrak{U} = \langle U \rangle$, $|\mathfrak{U}| = u$, and $|\mathfrak{P} : D(\mathfrak{P})| = p^n$. Let $\mathscr{L}$ be the Lie ring associated to $\mathfrak{P}$ ([12] p. 328). Then $\mathscr{L} = \mathscr{L}_1^* \oplus \mathscr{L}_2$ where $\mathscr{L}_1^*$ and $\mathscr{L}_2$ correspond to $\mathfrak{P}/\mathfrak{P}'$ and $\mathfrak{P}'$ respectively. Let $\mathscr{L}_1 = \mathscr{L}_1^*/p\mathscr{L}_1^*$. For $i = 1, 2$, let U_i be the linear transformation induced by U on $\mathscr{L}_i$.

LEMMA 6.1. *Assume that Hypothesis 6.1 is satisfied. Let $\varepsilon_1, \cdots, \varepsilon_n$ be the characteristic roots of U_1. Then the characteristic roots of U_2 are found among the elements $\varepsilon_i\varepsilon_j$ with $1 \leqq i < j \leqq n$.*

Proof. Suppose the field is extended so as to include $\varepsilon_1, \cdots, \varepsilon_n$. Since $\mathfrak{U}$ is a p'-group, it is possible to find a basis $x_1, \cdots, x_n$ of $\mathscr{L}_1$ such that $x_iU_1 = \varepsilon_i x_i$, $1 \leqq i \leqq n$. Therefore, $x_iU_1 \cdot x_jU_1 = \varepsilon_i\varepsilon_j x_i \cdot x_j$. As U induces an automorphism of $\mathscr{L}$, this yields that

$$(x_i \cdot x_j)U_2 = x_iU_1 \cdot x_jU_1 = \varepsilon_i\varepsilon_j x_i \cdot x_j .$$

Since the vectors $x_i \cdot x_j$ with $i < j$ span $\mathscr{L}_2$, the lemma follows.

By using a method which differs from that used below, M. Hall proved a variant of Lemma 6.2. We are indebted to him for showing us his proof.

LEMMA 6.2. *Assume that Hypothesis 6.1 is satisfied, and that U_1 acts irreducibly on $\mathscr{L}_1$. Assume further that $n = q$ is an odd prime and that U_1 and U_2 have the same characteristic polynomial. Then $q > 3$ and*

$$u < 3^{q/2}$$

Proof. Let ε^{p^i} be the characteristic roots of U_1, $0 \leqq i < n$. By Lemma 6.1 there exist integers i, j, k such that $\varepsilon^{p^i}\varepsilon^{p^j} = \varepsilon^{p^k}$. Raising this equation to a suitable power yields the existence of integers a and b with $0 \leqq a < b < q$ such that $\varepsilon^{p^a + p^b - 1} = 1$. By Hypothesis 6.1 (ii), the preceding equality implies $p^a + p^b - 1 \equiv 0 (\text{mod } u)$. Since U_1 acts irreducibly, we also have $p^q - 1 \equiv 0 (\text{mod } u)$. Since $\mathfrak{U}$ is a p'-group,