# ON ISOMETRIC ISOMORPHISM OF GROUP ALGEBRAS

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**1. Introduction.** Let $G$ be a locally compact group with right invariant Haar measure $m$ [2, Chapter XI]. The class $L(G)$ of integrable functions on $G$ forms a Banach algebra, with norm and product defined respectively by

$$\|x\| = \int |x(g)|\, m(dg)\ ,$$

$$(xy)(g) = \int x(gh^{-1})\, y(h)\, m(dh)\ .$$

The algebra is called real or complex according as the functions $x(g)$ and the scalar multipliers take real or complex values.

Suppose that $\tau$ is an isomorphism (algebraic and homeomorphic) of the group $G$ onto a second locally compact group $\Gamma$ having right invariant Haar measure $\mu$; let $c$ be the constant value of the ratio $m(E)/\mu(\tau E)$, and let $\chi$ be a continuous character on $G$. If $T$ is the mapping of $L(G)$ onto $L(\Gamma)$ defined by

$$(Tx)(\tau g) = c\,\chi(g)\; x(g), \qquad x \in L(G),$$

then it is easily verified that $T$ is a linear map preserving products and norms; for short, $T$ is an *isometric isomorphism* of $L(G)$ onto $L(\Gamma)$.

It is the purpose of the present note to show that, conversely, *any isometric isomorphism of* $L(G)$ *onto* $L(\Gamma)$ *has the above form,* in both the real and complex cases.

We mention in passing that if $T$ is merely required to be a *topological* isomorphism then $G$ and $\Gamma$ need not even be algebraically isomorphic. In fact, let $G$ and $\Gamma$ be any two finite abelian groups each having $n$ elements, of which $k$ are of order 2. Then the complex group algebras of $G$ and $\Gamma$ are topologically isomorphic to the direct sum of $n$ complex fields, and the real algebras are topologically isomorphic to the direct sum of $k+1$ real fields and $(n-k-1)/2$ two-dimensional algebras equivalent to the complex field. The algebraic content of this statement

Received October 24, 1950.

*Pacific J. Math.* 1 (1951), 305-311.