A THEOREM ON RINGS OF OPERATORS

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1. Introduction. The main result (Theorem 1) proved in this paper arose in connection with investigations on the structure of rings of operators. Because of its possible independent interest, it is being published separately.

The proof of Theorem 1 is closely modeled on the discussion in Chapter I of [3]. The connection can be briefly explained as follows. Let N be a factor of type II_1; then in addition to the usual topologies on N, we have the metric defined by $[[A]]^2 = T(A^*A)$, T being the trace on N. Now it is a fact that in any *bounded* subset of N, the $[[\ ]]$-metric coincides with the strong topology—this is the substance of Lemma 1.3.2 of [3]. In the light of this observation, it can be seen that Theorem 1 is essentially a generalization (to arbitrary rings of operators) of the ideas in Chapter I of [3].

Before stating Theorem 1, we collect some definitions for the reader's convenience. Let R be the algebra of all bounded operators on a Hilbert space H (of any dimension). In R we have a natural norm and *-operation. A typical neighborhood of 0 for the strong topology in R is given by specifying $\epsilon > 0$, $\xi_1, \cdots, \xi_n \in H$, and taking the set of all A in R with $\|A\xi_i\| < \epsilon$; for the weak topology we specify further vectors $\eta_1, \cdots, \eta_n \in H$ and take the set of all A with $|(A\xi_i, \eta_i)| < \epsilon$. By a *-algebra of operators we mean a self-adjoint subalgebra of R, that is, one containing A^* whenever it contains A; unless explicitly stated, it is not assumed to be closed in any particular topology. For convex subsets of R, and in particular for subalgebras, strong and weak closure coincide [2, Th. 5]. An operator A is self-adjoint if $A^* = A$, normal if $AA^* = A^*A$, unitary if $AA^* = A^*A =$ the identity operator I.

2. The main result. We shall establish the following result.

THEOREM 1. *Let M, N be *-algebras of operators on Hilbert space, $M \subset N$, and suppose M is strongly dense in N. Then the unit sphere of M is strongly dense in the unit sphere of N.*

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