RECURRENCE TIMES FOR THE EHRENFEST MODEL

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1. Introduction and summary. In 1907, P. and T. Ehrenfest [1] used a simple urn scheme as a pedagogic device to elucidate some apparent paradoxes in thermodynamic theory. Their model undergoes fluctuations intuitively related to fluctuations about equilibrium of certain thermodynamic systems. In view of an apparent discord among physicists [6, pp. 139-145] we shall not try to force an analogy with entropy.

The original Ehrenfest scheme was defined as follows. Initially, $2N$ balls are divided in an arbitrary manner between two urns, 1 and 2, the balls being numbered from 1 to $2N$. An integer between 1 and $2N$ is selected at random, each such integer having probability $(2N)^{-1}$, and the ball with the number selected is transferred from one urn to the other. The process is repeated any number of times. If n_1 and n_2 are the numbers of balls in urns 1 and 2 respectively before a transfer, it is clear that the probability is $n_1/(2N)$ that the transfer is from urn 1 to urn 2 and $n_2/(2N)$ that it is in the contrary direction.

Let $x'(n)$ be the number of balls in urn 1 after n transfers, and let $L'_{j,k}$ be be the smallest integer m such that $x'(m) = k$, given that $x'(0) = j$. If $k = j$, we call $L'_{k,k}$ the *recurrence time* for the state k. If $k \neq j$, we call $L'_{j,k}$ the *first-passage time* from j to k. The distribution of $x'(n)$, known classically, was derived by Kac [5] as an example of the use of matrix methods. Kac then found the mean and variance of $L'_{j,k}$, attributing some of his methods to Uhlenbeck. Friedman [4] found the moment-generating function for $x'(n)$ (for the Ehrenfest and more general models) by solving a difference-differential equation.

Instead of the original Ehrenfest model, we shall discuss a modified scheme with a continuous time parameter, which was apparently first suggested by A. J. F. Siegert [9]. In this scheme there are two urns and $2N$ balls initially divided between them arbitrarily. Each ball acts, independently of all the others, as follows: there is a probability of $(1/2)\,dt + o(dt)$ that the ball changes urns between t and $t + dt$, and a probability of $1 - [(1/2)\,dt + o(dt)]$ that the ball remains

Received August 5, 1950, and, in revised form, November 8, 1950.
Pacific J. Math. 1 (1951), 179-193.