A CHARACTERIZATION OF THE SUBGROUPS OF THE ADDITIVE RATIONALS

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1. Introduction. In the class of abelian groups every element of which (except the identity) has infinite order, the subgroups of the additive group of rational numbers have the simplest structure. These rational groups are the groups of rank one, or generalized cyclic groups, an abelian group G being said to have rank one if for any pair of elements, $a \neq 0$, $b \neq 0$, in G, there exist integers m, n, such that $ma = nb \neq 0$. Although many of the properties of these groups are known [1], it seems worthwhile to give a simple characterization from which their properties can easily be derived. This characterization is given in Theorems 1 and 2 of §2, and the properties of the rational groups are obtained as corollaries of these theorems in §3. In §4, all rings which have a rational group as additive group are determined.

Let $p_1, p_2, \cdots, p_j, \cdots$ be an enumeration of the primes in their natural order; and associate with each p_j an exponent k_j, where k_j is a nonnegative integer or the symbol ∞. We consider sequences i; $k_1, k_2, \cdots, k_j, \cdots$, where i is any positive integer such that $(i, p_j) = 1$ if $k_j > 0$, and define $(i; k_1, k_2, \cdots, k_j, \cdots) = (i; k_j)$ to be the set of all rational numbers of the form ai/b, where a is any integer and b is an integer such that $b = \prod'_{p_j} p_j^{n_j}$ with $n_j \leq k_j$. Then each sequence determines a well-defined set of rational numbers. The symbol $\prod'$ designates a product over an arbitrary subset of the primes that satisfy whatever conditions are put on them; $\prod$ designates a product over all primes that satisfy the given conditions.

2. Characterization of the rational groups. We show that the nontrivial subgroups of R are exactly the subsets $(i; k_j)$ defined in the introduction.

THEOREM 1. *The set $(i; k_j)$ is a subgroup of R^+, the additive group of rational numbers. We have $(i; k_j) = (i'; k'_j)$ if and only if $i = i'$, $k_j = k'_j$ for all j.*

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