ON THE LERCH ZETA FUNCTION

T. M. APOSTOL

1. Introduction. The function $\phi(x, a, s)$, defined for $\Re s > 1$, x real, $a \neq$ negative integer or zero, by the series

$$\phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(a+n)^s}, \tag{1.1}$$

was investigated by Lipschitz [4; 5], and Lerch [3]. By use of the classic method of Riemann, $\phi(x, a, s)$ can be extended to the whole s-plane by means of the contour integral

$$I(x, a, s) = \frac{1}{2\pi i} \int_C \frac{z^{s-1} e^{az}}{1 - e^{z+2\pi i x}} \, dz, \tag{1.2}$$

where the path C is a loop which begins at $-\infty$, encircles the origin once in the positive direction, and returns to $-\infty$. Since $I(x, a, s)$ is an entire function of s, and we have

$$\phi(x, a, s) = \Gamma(1-s) I(x, a, s), \tag{1.3}$$

this equation provides the analytic continuation of ϕ. For integer values of x, $\phi(x, a, s)$ is a meromorphic function (the Hurwitz zeta function) with only a simple pole at $s = 1$. For nonintegral x it becomes an entire function of s. For $0 < x < 1$, $0 < a < 1$, we have the functional equation

$$\phi(x, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{\pi i (s/2 - 2ax)} \phi(-a, x, s) + e^{\pi i (-s/2 + 2a(1-x))} \phi(a, 1-x, s) \right\}, \tag{1.4}$$

first given by Lerch, whose proof follows the lines of the first Riemann proof of the functional equation for $\zeta(s)$ and uses Cauchy's theorem in connection with the contour integral (1.2).

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