TWO THEOREMS ON METRIC SPACES

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1. Introduction. Let E be a metric space with distance function d. The space E is called *two-point homogeneous* if given any four points a, a', b, b' with $d(a, a') = d(b, b')$, there exists an isometry of E carrying a, a' to b, b', respectively. In a recent paper [7], the author has determined all the compact and connected two-point homogeneous spaces. It is the aim of the present note to discuss the noncompact case, and prove a conjecture of Busemann which can be regarded also as a sharpening of a theorem of Birkhoff [1]. The results concerning the noncompact two-point homogeneous spaces are not as satisfactory as the results for the compact case; we have to assume certain conditions on the metric.

By a segment in a metric space E, we shall mean an isometric image of a closed interval with the usual metric. A metric space will be said to have the property (L) if given a point p, there exists a neighborhood W of p so that each point x $(\neq p)$ of W can be joined to p by at most one segment in E. The following theorems will be proved:

THEOREM 1. *Let E be a finite-dimensional, finitely compact, convex metric space with property (L). If E is two-point homogeneous, then E is homeomorphic with a manifold.*

THEOREM 2. *Let E be a metric space with all the properties mentioned in Theorem* 1. *If, moreover,* $\dim E$ *is odd, then E is congruent either to the euclidean space, the hyperbolic space, the elliptic space, or the spherical space.*

Our Theorem 2 justifies the conjecture of Busemann [2, p. 233] that a two-point homogeneous three dimensional S.L. space [2, p. 78] is either elliptic, hyperbolic, or euclidean. It is to be noted that Theorem 2 no longer holds if $\dim E$ is even and greater than two. The complex elliptic spaces [7] and the hyperbolic Hermitian spaces[1] [2, p. 192] serve as counter examples.

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[1]These spaces were first introduced by H. Poincaré, and then discussed by G. Fubini and E. Study. Following E. Cartan, we call these spaces the hyperbolic Hermitian spaces.

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