A NOTE ON UNRESTRICTED REGULAR TRANSFORMATIONS

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1. Introduction. Let W be the class of real continuous functions defined on the nonnegative reals and such that for each $g(t) \in W$ the following conditions hold:

(a) $g(0) = 0$ and $g(t) > 0$ when $t > 0$,

(b) for each triple $t_1, t_2, t_3, \geq 0$, the inequality $t_1 + t_2 \geq t_3$ implies $g(t_1) + g(t_2) \geq g(t_3)$.

Let M be a metric space wherein $[p, q]$ denotes the distance between $p, q \in M$. A transformation $T(M) = N$ is called *unrestricted regular* by W. A. Wilson [2] if there exists a $g(t) \in W$ such that for each pair $p, q \in M$ we have $[T(p), T(q)] = g[p, q] \equiv g([p, q])$. The function g (not always unique) is called a *scale function* for T.

It is easily seen that every member of the class W is monotone increasing and that each unrestricted regular transformation is continuous and one-to-one. Thus an unrestricted regular transformation on a compact metric space is a homeomorphism. Wilson shows [2, p.65] that if M is dense and metric and T is unrestricted regular, then T is a homeomorphism.

In §2 of this note we examine the graphs of scale functions and show how the graph of the scale function of an unrestricted regular transformation determines the behavior of points under the transformation. Section 3 is devoted to a question involving a class of transformations investigated by E. J. Mickle [1].

2. The graphs of scale functions. We shall establish the following result.

THEOREM 1. *If M is a metric space and $T(M) = M$ is unrestricted regular with scale function $g(t)$, then for each $n = 1, 2, 3, \cdots$, the transformation $T^n(M) = M$ is unrestricted regular with scale function $g^n(t)$ (that is, g iterated n times).*

Proof. Obviously $g^n(t)$ is real and continuous, $g^n(0) = 0$, and $g^n(t) > 0$ when $t > 0$. Suppose $T^{n-1}(M) = M$ is unrestricted regular with scale function $g^{n-1}(t)$.

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