ON THE THEORY OF SPACES Λ

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1. Introduction. In this paper we discuss properties of the spaces $\Lambda(\phi, p)$, which were defined for the special case $\phi(x) = \alpha x^{\alpha-1}$, $0 < \alpha \leq 1$, in our previous paper [8]. A function $f(x)$, measurable on the interval $(0, l)$, $l < +\infty$ belongs to the class $\Lambda(\phi, p)$ provided the norm $\|f\|$, defined by

$$(1.1) \qquad \|f\| \equiv \left\{\int_0^l \phi(x)\, f^*(x)^p dx\right\}^{1/p},$$

is finite. Here $\phi(x)$ is a given nonnegative integrable function on $(0, l)$, not identically 0, and $f^*(x)$ is the decreasing rearrangement of $|f(x)|$, that is, the decreasing function on $(0, l)$, equimeasurable with $|f(x)|$. (For the properties of decreasing rearrangements see [5, 12, 7, and 8].) We write also $\Lambda(\alpha, p)$ instead of $\Lambda(\phi, p)$ with $\phi(x) = \alpha x^{\alpha-1}$, and $\Lambda(\phi)$ instead of $\Lambda(\phi, 1)$. We shall also consider spaces $\Lambda(\phi, p)$ for the infinite interval $(0, +\infty)$. In §2 we give some simple properties of the spaces Λ, and show in particular that $\Lambda(\phi, p)$ has the triangle property if and only if $\phi(x)$ is decreasing. In §3 we discuss the conjugate spaces $\Lambda^*(\phi, p)$, and show that the spaces $\Lambda(\phi, p)$ are reflexive. In §4 we give a generalization of the spaces $\Lambda(\phi, p)$, and characterize the conjugate spaces in case $p = 1$. In §5 we give applications; we prove that the Hardy-Littlewood majorants $\theta(x, f)$ of a function $f \in \Lambda(\phi, p)$ or $f \in \Lambda^*(\phi, p)$ also belong to the same class. We give sufficient conditions for an integral transformation to be a linear operation from one of these spaces into itself, and apply them to solve the moment problem for the spaces $\Lambda(\phi, p)$ and $\Lambda^*(\phi, p)$.

2. Properties of spaces $\Lambda(\phi, p)$. We shall establish the following result.

THEOREM 1. *The norm $\|f\|$ defined by* (1.1) *has the triangle property if and only if $\phi(x)$ is equivalent to a decreasing function; in this case $f, g \in \Lambda(\phi, p)$*

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