AN EXTENSION OF TIETZE'S THEOREM

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1. Introduction. Let X be an arbitrary metric space, A a closed subset of X, and E^n the Euclidean n-space. Tietze's theorem asserts that any (continuous) $f: A \longrightarrow E^1$ can be extended to a (continuous) $F: X \longrightarrow E^1$. This theorem trivially implies that any $f: A \longrightarrow E^n$ and any $f: A \longrightarrow$ (Hilbert cube) can be extended; we merely decompose f into its coordinate mappings and observe that, in these cases, the continuity of each of the coordinate mappings is equivalent to that of the resultant map.

Where this equivalence is not true, for example mapping into the Hilbert space, the theorem has been neglected. We are going to prove that, in fact, Tietze's theorem is valid for continuous mappings of A into any locally convex linear space (4.1), (4.3). Two proofs of this result will be given; the second proof (4.3), although essentially the same as the first, is more direct; but it hides the geometrical motivation.

There are several immediate consequences of the above result. First we obtain a theorem on the simultaneous extension of continuous real-valued functions on a closed subset of a metric space (5.1). Secondly, we characterize completely those normed linear (not necessarily complete) spaces in which the Brouwer fixed-point theorem is true for their unit spheres (6.3). Finally, we can generalize the whole theory of locally connected spaces to arbitrary metric spaces. By way of illustration, we prove a theorem about absolute neighborhood retracts that is apparently new even in the separable metric case (7.5).

The idea of the proof of the main theorem is simple. Given A and X, we show how to replace $X - A$ by an infinite polytope; we extend f continuously first on the vertices of the polytope, and then over the entire polytope by linearity. For this we need several preliminary remarks on coverings and on polytopes.

2. On coverings and polytopes. If X is any space, a covering of X by an arbitrary collection $\{U\}$ of open sets is called a *locally finite covering* if, given any

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