MATRICES OF QUATERNIONS

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1. Introduction. In this note, some theorems which concern matrices of complex numbers are generalized to matrices over real quaternions. First it is proved that every matrix of quaternions has a characteristic root. Next, there exist $n - 1$ mutually orthogonal unit n-vectors all orthogonal to a given vector. It is shown that Schur's lemma holds for matrices of quarternions: every matrix can be transformed into triangular form by a unitary matrix. For individual quaternions, it is known that two quaternions are similar if they have the same trace and the same norm—thus every quaternion has a conjugate $a + bj(b \geq 0)$. This fact is proved again.

The quaternion λ is called a *characteristic root* of a (square) matrix A provided a non-zero vector x exists such that $Ax = x\lambda$. Similar matrices have the same characteristic roots; if $y = Tx$, where T has an inverse, then $TAT^{-1}y = TAx = Tx\lambda = y\lambda$. Another interesting fact is that if λ is a characteristic root, then so is $\rho^{-1}\lambda\rho$; for from $Ax = x\lambda$ follows $A(x\rho) = (x\rho)\rho^{-1}\lambda\rho$; thus if the vector corresponding to the characteristic root λ is x, then $x\rho$ is the vector corresponding to the characteristic root $\rho^{-1}\lambda\rho$.

2. Lemma. We shall need the following result.

LEMMA 1. *If $A = (a_{i,j})$ is a matrix of elements from any field or fields, then a triangular matrix T exists such that $T^{-1}AT = C = (c_{i,j})$, where $c_{i,j} = 0$ whenever $i > j + 1$. The elements of T are rational functions of the elements of A.*

Proof. The proof consists in transforming A in steps so that an additional zero appears at each step. First A is transformed so that all the elements in the first column (except the first two) become zero; the transformed matrix is further transformed so that all the elements in the second column (except the first three) become zero, and so on. The formal proof is inductive; it will be sufficient to give the idea of the proof. In the first column of A, either $a_{j,1} = 0$ for all $j > 1$, or else

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