APPLICATIONS OF THE THEORY OF QUADRATIC FORMS IN HILBERT SPACE TO THE CALCULUS OF VARIATIONS

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1. Introduction. One of the interesting chapters in the calculus of variations is the theory of indices. It has two aspects, the theory in the large and the theory in the small. An important part of the latter is the theory of indices of the second variation, that is, of integrals that are quadratic in their arguments. Such a theory includes the Sturm-Liouville theory for self-adjoint differential systems. The theory of the second variation can be approached from many points of view. It can be developed by means of the theory of differential equations and the associated boundary value problems. A description of this method together with references to the many writers on this subject can be found in the works of Bliss [3; 4] and Reid [14; 15]. The first of these papers by Reid is an excellent introduction to the present paper; in it is found an extensive bibliography on boundary value problems together with a description of various methods of studying such problems. The second variation can also be studied by the use of the theory of broken extremals, as has been done by Morse [13]. A third method is by means of "natural isoperimetric conditions"; this was done by Birkhoff and Hestenes [2] and also by Hazard [7], Karush [10], and Ritcey [17].

The author has been convinced for some time that the theory of the second variation can be obtained from an appropriate theory of quadratic forms in Hilbert space. The purpose of the present paper is to show how this can be done. The theorems in Hilbert space which we shall use can be found in standard references on Hilbert space [18; 19]. In order to apply these results to the calculus of variations, it is more convenient to emphasize the quadratic form, itself, instead of the self-adjoint transformation associated with the quadratic form. For this reason a portion of the paper is devoted to rephrasing known results concerning transformations in terms of quadratic forms.

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