QUASI-CONVEXITY AND THE LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

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1. Introduction. We are concerned in this paper with integrals of the form

$$(1.1) \qquad I(z,D) = \int_D f\left[x, z^i(x),\ z^i_{x^\alpha}(x)\right] dx,$$

where

$$x = (x^1, \cdots, x^\nu),\ z = (z^1, \cdots, z^N),\ p = p^i_\alpha$$

$$(i = 1, \cdots, N;\ \alpha = 1, \cdots, \nu),$$

$f(x, z, p)$ is continuous in its arguments, and D is a bounded domain.

The object of the paper is to discuss necessary and sufficient conditions on the function f for the integral I to be lower semicontinuous with respect to various types of convergence of the vector functions z. Because of the success of the "direct methods" in the Calculus of Variations, many writers have shown that certain integrals are lower semicontinuous. However, the writer knows of no paper in which a *necessary* condition for lower semicontinuity was discussed, although such a condition is very easy to obtain (see Theorem 2.1).

In §2, a general condition called "quasi-convexity" (see Definition 2.2) on the behavior of f as a function of p is obtained which is both necessary and sufficient for the lower semicontinuity of I with respect to the type of convergence given in Definition 2.1. This condition is that any linear function furnish the absolute minimum to $I(z,D)$ among all Lipschitzian (see below) functions which coincide with it on D^*, D being any bounded domain and D^* its boundary; here, of course, we consider f to be a function of p only. Section 3 discusses cases involving more general types of convergence and gives an existence theorem. In §4, it is shown that if $f(p)$ is continuous and quasi-convex, then it satisfies a certain generalized Weierstrass condition which reduces to the ordinary one (for the case at hand) when f is of class C'; this is, in turn, seen to be equivalent to the Legendre-Hadamard condition (see (4.8)) (quasi-regularity in its general form) when f is of class C . In §5, a general sufficient

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