SYMMETRIC PERPENDICULARITY IN HILBERT GEOMETRIES

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1. **Introduction.** A hilbert plane geometry [2] can be generated in the following way. Let K be a simple, closed, convex curve in the euclidean plane and H its open interior. If a and b are any two points in H, they determine a line $a \times b$[1] which intersects K in a pair of points u and v. With R denoting cross-ratio, the hilbert distance from a to b is defined by

$$h(a, b) = k \ |\log R(a, b; u, v)|,$$

where k is an arbitrary positive constant. The region H is then a metric set with respect to K. Under the additional requirement that K contain at most one segment, H defines a hilbert plane geometry in which any pair of points are uniquely connected by a geodesic, and these geodesics are open straight lines. If K is an ellipse, then the hilbert geometry coincides with the well-known Klein model of hyperbolic geometry.

Perpendicularity in H is defined through the idea of distance. If p and ξ are any point and line respectively, then a point f on ξ is a "foot of p on ξ" if $h(p, f) \leq h(p, x)$ for all points x on ξ. A line η, intersecting ξ, is perpendicular to ξ if every point on η has the point of intersection, $\xi \times \eta$, as a foot on ξ. Under this definition, there is no need for the perpendicularity of η to ξ to imply the perpendicularity of ξ to η. The aim here is to show that when perpendicularity is always symmetric, the hilbert geometry is hyperbolic.

As before, let p and ξ be any point and line in H, and let η be a line passing through p and intersecting K in the points u and v. It can be shown quite simply that a necessary and sufficient condition for η to be perpendicular to ξ is that a pair of supporting lines exist, one at u and one at v, intersecting at a point w on ξ [1]. If η is perpendicular to ξ, then the previous statement implies that η is also perpendicular to every line through w which is a secant to K. When such a secant cuts K at points m and n, then symmetry of perpendicularity requires that a supporting line exist at m, and one at n, such that the two intersect on η.

[1] Here and henceforth the line joining a and b will be indicated by $a \times b$, and symmetrically the point of intersection on lines ξ and η by $\xi \times \eta$.

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