A NOTE ON THE DIMENSION THEORY OF RINGS

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1. Introduction. Let O be an integral domain. If in O there is a proper chain

$$(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset (1)$$

of prime ideals, but no such chain

$$(0) \subset P_1' \subset \cdots \subset P_{n+1}' \subset (1),$$

then O will be said to be *n-dimensional*. Let O be of dimension n: the question is whether the polynomial ring $O[x]$ is necessarily $(n+1)$-dimensional. Here, as throughout, x is an indeterminate.

By an *F-ring* we shall mean a 1-dimensional ring O such that $O[x]$ is not 2-dimensional (i. e., the proposed assertion that $O[x]$ is necessarily 2-dimensional fails). Given an F-ring, we try by definite constructions to pass to a larger F-ring having the same quotient field: this restricts the class of rings in which to look for an F-ring—a priori we do not know they exist. In this way we also come (in Theorem 8 below) to a complete characterization of F-rings: if O is 1-dimensional, then $O[x]$ is 2-dimensional if and only if every quotient ring of $\bar{O}$, the integral closure of O, is a valuation ring. The rings $\bar{O}$ thus coincide (for dimension 1) with Krull's Multiplikationsringe [5; p. 554].

2. Preliminary results. The first five theorems are of a preparatory character, and the proofs offer no difficulties.

THEOREM 1. *Let O be an arbitrary commutative ring with 1, P_1, P_2, P_3 distinct ideals in $O[x]$. If $P_1 \subset P_2 \subset P_3$, and P_2 and P_3 are prime ideals, then P_1, P_2, P_3 cannot have the same contraction to O.*

Proof. Let

$$P_1 \cap O = P_2 \cap O = p,$$

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