LIPSCHITZ FUNCTIONS OF CONTINUOUS FUNCTIONS

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1. Introduction. The present paper was suggested by a note of W. S. Loud [2] in which the following theorem on functions of a real variable is proved.

THEOREM 1. *If* α *is a constant* $(0 < \alpha < 1)$, *there exist a continuous function* $f(t)$ *and a pair of positive constants* K_1 *and* K_2 *such that*

$$|f(t+h) - f(t)| < K_1 \; |h|^{\alpha}$$

for all t and all h, and such that

$$\limsup_{h \to 0} \frac{|f(t+h) - f(t)|}{|h|^{\alpha}} > K_2$$

for all t.

It is natural to examine the possibility of a variable exponent $\alpha(t)$ and to consider various definitions that associate with every continuous function $f(t)$ a "Lipschitz function" $\alpha(t; f)$. For a reasonable choice of the definition, Loud's result implies that every constant $\alpha\,(0 < \alpha < 1)$ is the Lipschitz function of some continuous function. The following sections offer two different definitions of Lipschitz functions, and deal with the problem of characterizing the functions that are Lipschitz functions of continuous functions.

2. The point Lipschitz function of a function. Let $f(t)$ be a continuous, real-valued function of t. Consider the quantity

$$Q(\alpha, t_0; f) = \limsup_{h \to 0} \frac{|f(t_0+h) - f(t_0)|}{|h|^{\alpha}}.$$

If $Q(\alpha, t_0; f)$ is finite for $\alpha = \alpha'$, it is zero for all α less than α'; if Q is greater than zero for $\alpha = \alpha'$, it has the value $+\infty$ for all α greater than α'. Let $\alpha(t_0; f)$ denote the least upper bound of all α for which $Q(\alpha, t_0; f)$ is

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