AN ISOPERIMETRIC MINIMAX

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Introduction. In the preceding paper J. W. Green considers for a given convex body K in the euclidean plane the minimum of the isoperimetric ratio r (ratio of squared perimeter l^2 to area a) taken over all affine transforms k of K. He then investigates the maximum value taken over all K of this minimum ratio, shows by variational methods that such a maximum is attained by some polygon of five or fewer sides, and conjectures that it is, in fact, attained by a triangle with $12\sqrt{3}$, the isoperimetric ratio of an equilateral triangle, as the minimax ratio. I shall prove this conjecture directly by refining an estimation used by Green, the precise statement of results being as follows:

I. *Let K be an nontriangular plane convex body; there then exists an affine transform k of K with $r(k) < 12\sqrt{3}$.*

II. *Let T be a nonequilateral triangle; then $r(T) > 12\sqrt{3}$.*

Before taking up the proof of these results we dispose of a lemma.

III. *Let k be a possibly degenerate convex body with $s \subset k \subset t$, wherein t is an equilateral triangle, and s a side of t; there then exists a number x with $0 \leqq x \leqq 1$ such that*

$$l(k) \leqq (2/3 + x/3)\ l(t)$$

$$a(k) \geqq x\ a(t),$$

simultaneous equality occurring if and only if either $x = 0$, $k = s$ or $x = 1$, $k = t$.

Proof of III. Let p be that supporting strip of k parallel to the line-segment s; and let x be the ratio of the width of p to the width or altitude of t. Thus $0 \leqq x \leqq 1$, with $x = 0$ or $x = 1$ according as $k = s$ or $k = t$. Choose a point at which k touches the side of p opposite s, and define k_* to be the triangle with this point as apex and s as base. Define k^* to be the trapezoid formed by intersection of p and t. Clearly $s \subset k_* \subset k \subset k^* \subset t$; and $k_* = k = k^*$ if and only if $k = s$ or $k = t$.

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