SOME THEOREMS ON THE SCHUR DERIVATIVE

L. CARLITZ

1. Introduction. Given the sequence $\{a_m\}$ and $p \neq 0$, Schur [5] defined the derivative a_m' by

$$(1.1) \qquad a_m' = \Delta a_m = (a_{m+1} - a_m)/p^{m+1};$$

higher derivatives are defined by means of

$$a_m^{(r)} = \Delta^r a_m = \Delta(a_m^{(r-1)}), \quad a_m^{(0)} = a_m.$$

In particular if p is a prime, a an integer and $a_m = a^{p^m}$, then by Fermat's theorem

$$a_m' = (a^{p^{m+1}} - a^{p^m})/p^{m+1}$$

is integral. Schur proved that if $p \nmid a$, then also the derivatives

$$\Delta^2 a^{p^m}, \ \Delta^3 a^{p^m}, \cdots, \ \Delta^{p-1} a^{p^m}$$

are all integral. Moreover if $a_0' \equiv 0 \pmod p$ then all the derivatives $\Delta^r a^{p^m}$ are integral, while if $a_0' \not\equiv 0 \pmod p$ then every number of $\Delta^p a^{p^m}$ has the denominator p.

A. Brauer [1] gave another proof of Schur's results. About the same time Zorn [6] proved these results by p-adic methods and indeed proved the following stronger theorem. For $x \equiv 1 \pmod p$, define

$$X_m = (x^{p^m} - 1)/p^{m+1},$$

and as above let $\Delta^r X_m$ denote the r-th derivative of X_m; then

$$(1.2) \qquad \Delta^r X_m \equiv \frac{(p-1)(p^2-1)\cdots(p^r-1)}{(r+1)!} X_m^{r+1} \pmod{p^m}$$

provided $r < p$; for $r < p-2$, the congruence (1.2) holds $(\bmod\ p^{m+1})$. It is also shown that Schur's theorem is an easy consequence of Zorn's results.

Received November 16, 1951.
Pacific J. Math. 3 (1953), 321-332