EXTENSION OF A RENEWAL THEOREM

DAVID BLACKWELL

1. Introduction. A chance variable x will be called a *d-lattice variable* if

$$\sum_{n=-\infty}^{\infty} \Pr\{x = nd\} = 1 \tag{1}$$

and

$$d \text{ is the largest number for which (1) holds.} \tag{2}$$

If x is not a d-lattice variable for any d, x will be called a *nonlattice variable.* The main purpose of this paper is to give a proof of:

THEOREM 1. *Let $x_1, x_2, \cdots$ be independent identically distributed chance variables with $E(x_1) = m > 0$ (the case $m = +\infty$ is not excluded); let $S_n = x_1 + \cdots + x_n$; and, for any $h > 0$, let $U(a, h)$ be the expected number of integers $n \geq 0$ for which $a \leq S_n < a + h$. If the x_n are nonlattice variables, then*

$$U(a, h) \longrightarrow \frac{h}{m}, 0 \qquad \textit{as } a \longrightarrow +\infty, -\infty.$$

If the x_n are d-lattice variables, then

$$U(a, d) \longrightarrow \frac{d}{m}, 0 \qquad \textit{as } a \longrightarrow +\infty, -\infty.$$

(*If $m = +\infty$, h/m and d/m are interpreted as zero.*)

This theorem has been proved (A) for nonnegative d-lattice variables by Kolmogorov [5] and by Erdös, Feller, and Pollard [4]; (B) for nonnegative nonlattice variables by the writer [1], using the methods of [4]; (C) for d-lattice variables by Chung and Wolfowitz [3]; (D) for nonlattice variables such that the distribution of some S_n has an absolutely continuous part and $m < \infty$ by Chung

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