CONVEXITY PROPERTIES OF INTEGRAL MEANS OF ANALYTIC FUNCTIONS

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1. Introduction. Let $f = f(z)$ denote an analytic function of the complex variable z in the open circle $|z| < R$. For each positive number t, the mean of order t of the modulus of $f(z)$ is defined as follows:

$$\mathfrak{M}_t(r; f) = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^t \, d\theta\right]^{1/t}, \qquad (0 \leq r < R).$$

The reader might consult [5, p. 143-144; 3; and 4, p. 134-146] for some of the properties of this mean value function $\mathfrak{M}_t(r; f)$.

We consider the question: does the analyticity in $|z| < R$ of the function f imply the convexity of the mean $\mathfrak{M}_t(r; f)$ as a function of r in the interval $0 \leq r < R$? It is known [1] that:

(A) Unless the function f is suitably restricted, the set of positive values t for which the question may be answered affirmatively has a finite upper bound.

(B) If the number t is of the form $2/k$, with k a positive integer, then, for every analytic function f, the mean of order t is convex.

(C) If the function f vanishes at the origin, then the mean $\mathfrak{M}_t(r; f)$ is convex for every fixed positive number t.

(D) If the function f has no zero in the circle, then its mean of order t is convex, provided that the positive number satisfies $t \leq 2$.

(E) If the function f has at most k zeros, $k \geqq 1$, in the circle, then the mean of order t is convex provided that the positive number t satisfies $t \leqq 2/k$.

The main purpose of this paper is to prove that, for every analytic function f, the mean of order four is convex. Moreover, we show by example that if the number t is greater than 5.66, then there is an analytic function whose mean of order t is not convex.

2. Means of nonvanishing functions. Assume that $g(z)$ is analytic in $|z| < R$,

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