ITERATES OF ARITHMETIC FUNCTIONS AND A PROPERTY OF THE SEQUENCE OF PRIMES

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1. Introduction. In a previous paper [2], the author has investigated certain properties of the iterates of arithmetic functions which are of the following form. For $n = \prod p_i^{\alpha_1}$,

$$f(n) = \prod f(p_i)\,[A(p_i)]^{\alpha_i - 1}, \tag{1.1}$$

where $f(p_i)$ is an integer, $1 < f(p_i) < p_i$, and $A(p_i)$ is an integer $\leq p_i$, for odd primes p_i; whereas $f(2) = 1$, $A(2) = 2$. We shall denote the set of these arithmetic functions by K. These conditions ensure that for $n > 2$, $f(n) < n$, and hence if $f^k(n)$ denotes the k-th iterate of f there is a unique integer k such that

$$f^k(n) = 2. \tag{1.2}$$

For this k we write $k = C_f(n)$. We define

$$C_f(1) = C_f(2) = 0.$$

In this paper we propose to consider the problem of determining a $g \in K$ such that for all odd primes p, and all $f \in K$,

$$C_g(p) \geq C_f(p). \tag{1.3}$$

The solution to this problem produces an interesting property of the sequence of primes in that we shall show that (1.3) is equivalent to having g skip down through the primes. More precisely, if $p_1 = 2$, $p_2 = 3, \cdots$, and in general p_i denotes the i-th prime, (1.3) is equivalent to having $g(3) = 2$, $g(5) = 4$ or 3, and

$$g(p_i) = p_{i-1} \qquad \text{for } i > 3. \tag{1.4}$$

2. A theorem concerning functions of K. In carrying out the proof of the result

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