ON THE LINEAR INDEPENDENCE OF ALGEBRAIC NUMBERS

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1. Introduction. Besicovitch [1] has proved by elementary methods involving only the concept of the irreducibility of equations the following:

THEOREM. *Let*

$$a_1 = b_1 p_1, \ a_2 = b_2 p_2, \cdots, a_s = b_s p_s,$$

where $p_1, p_2, \cdots, p_s$ *are different primes, and* $b_1, b_2, \cdots, b_s$ *are positive integers not divisible by any of these primes. If* $x_1, x_2, \cdots, x_s$ *are positive real roots of the equations*

$$x^{n_1} - a_1 = 0, \ x^{n_2} - a_2 = 0, \cdots, x^{n_s} - a_s = 0,$$

and $P(x_1, x_2. \cdots, x_s)$ *is a polynomial with rational coefficients of degree less than or equal to* $n_1 - 1$ *with respect to* x_1, *less than or equal to* $n_2 - 1$ *with respect to* x_2, *and so on, then* $P(x_1 x_2, \cdots, x_s)$ *can vanish only if all its coefficients vanish.*

It is rather surprising that this has not been proved before, since results of this kind occur as particular cases of a general investigation in the theory of algebraic numbers, and some have been known for many years. We have the well-known general problem:

PROBLEM. Let K be an algebraic number field, and let $x_1, x_2, \cdots, x_s$ be algebraic numbers of degrees $n_1, n_2, \cdots, n_s$ over K. When does the field $K(x_1, x_2, \cdots, x_s)$ have degree $n_1 n_2 \cdots n_s$ over K?

This holds if either the degrees or the discriminants over K of the fields $K(x_1), K(x_2), \cdots, K(x_s)$ are relatively prime in pairs. The first part is a simple consequence of the usual theory of reducibility when $s = 2$, and the extension is obvious. The second part for $s = 2$ is given as Theorem 87 in Hilbert's report on algebraic number fields, and its proof depends on algebraic number

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