ORTHOGONAL HARMONIC POLYNOMIALS

P. R. GARABEDIAN

1. Introduction. In this paper we develop sets of harmonic polynomials in x, y, z which are orthogonal over prolate and oblate spheroids. The orthogonality is taken in several different norms, each of which leads to the discussion of a partial differential equation by means of the kernel of the orthogonal system corresponding to that norm. The principal point of interest is that the orthogonality of the harmonic polynomials in question does not depend on the shape of the spheroids, but only on their size. More precisely, the polynomials depend only on the location of the foci of the ellipse generating the spheroid, and not on its eccentricity.

The importance of constructing these polynomials stems from the role which they play in the calculation of the kernel functions and Green's functions of the Laplace and biharmonic equations in a spheroid. One can compute from the kernels, in turn, the solution of the basic boundary-value problems for these equations. As a particular case, one arrives at formulas for the solution of the partial differential equation

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial z^2} = 0$$

which arises in discussion of axially symmetric flow.

Results of the type presented here have occurred previously in the work of Zaremba [**10**], and are related to recent developments of Friedrichs [**3**, **4**] and the author [**5**]. The polynomials investigated in this earlier work are in two independent real variables and yield formulas for solving the Laplace and biharmonic equations in two dimensions. Thus it is natural to suggest that the basic results generalize to n-dimensional space. In this connection, it is

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