THE NEUMANN PROBLEM FOR THE HEAT EQUATION

W. FULKS

1. Introduction. By the Neumann problem we mean the following boundary-value problem: to determine the solution $u(x, t)$ of the equation

$$(1.1) \qquad u_{xx}(x, t) - u_t(x, t) = 0$$

in the rectangle or semi-infinite strip $R^{(b,c)}$: $\{ b < x < c;\ a < t < T \leq \infty \}$, given $u(x, a)$ on $b < x < c$ and $u_x(b, t)$ and $u_x(c, t)$ on $a < t < T$. There is a formula in terms of the Green's function (essentially given by Doetsch in [2, p. 361]) which gives the answer to this problem if the closed rectangle is in the interior of a larger region in which $u(x, t)$ is a continuous solution of (1.1). This formula is as follows: let $d = c - b$, and let

$$F^{(b,c)}(x, t; y, s) = \frac{1}{2d}\left[\vartheta_3\left(\frac{x-y}{2d}, \frac{t-s}{d^2}\right) + \vartheta_3\left(\frac{x+y-2b}{2d}, \frac{t-s}{d^2}\right)\right]$$

where ϑ_3 is the Jacobi Theta function; then

$$(1.2) \quad u(x, t) = \int_b^c F^{(b,c)}(x, t; y, a)\, u(y, a)\, dy - \int_a^t F^{(b,c)}(x, t; b, s)\, u_x(b, s)\, ds$$
$$+ \int_a^t F^{(b,c)}(x, t; c, s)\, u_x(c, s)\, ds.$$

The purpose of this paper is to extend the use of formula (1.2) in the following manner: we will give conditions under which a solution of the heat equation can be written in the form (1.2) wherein $u(a, y)\, dy$, etc., are replaced by $dA(y)$ or by $a(y)\, dy$, where $A(y) \in \mathrm{BV}$ (that is, of bounded variation) or $a(y) \in L$. And we will examine the senses in which these extensions of formula (1.2) solve the boundary-value problem; that is, the manner in which the solutions tend to the prescribed boundary data for approach to a boundary point. Furthermore, we will obtain criteria for the unique determination of the solutions of these generalized boundary-value problems.

Received June 30, 1952. The work on this paper was performed under sponsorship of the Office of Naval Research, Contract Nonr-386(00); (NR 044 004).

Pacific J. Math. 3 (1953), 567-583