DERIVATIVES OF INFINITE ORDER

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1. Introduction. The major purpose here is to reexamine, chiefly from the standpoint of summation by Borel's exponential means, a number of problems concerning the existence and form of

$$\lim_{n \to \infty} f^{(n)}(x),$$

for x a real variable in an interval. Several articles have been contributed on this topic [**5, 6, 11, 16**], all of which take the limit process involved to be ordinary convergence. In one [**5**], however, Boas and Chandrasekharan point to the desirability of interpreting the limit process in a more general sense and state without proof that one of their results (the case $\alpha = 1$, $\lambda_n = 1$ for all n, of Theorem 4 below) can be established by their method for any (presumably linear) summation method T having the property that, as $n \longrightarrow \infty$,

(1) T-lim s_n exists and equals s implies T-lim s_{n-1} exists and equals s.

Borel's method of exponential means, like his integral method, possesses property (1) although, curiously, not its converse, as Hardy [cf. **9**, pp. 183, 196] pointed out. Methods satisfying both (1) and its converse include ordinary convergence and the summation methods of Abel, Cesàro, Euler, Hölder, and, when regular (see below), Voronoi-Nörlund.

It is not clear from [**5**] just how their proof of the cited result (that $f^{(n)}(x) \longrightarrow g(x)$ dominatedly in (a, b) implies $g(x) = ke^x$) can really be carried over to *all* linear summation methods of type (1). Since the transform $\{F_m(x)\}$, m discrete or continuous, of the sequence $\{f^{(n)}(x)\}$ converges dominatedly, it follows that

$$\lim_{m \to \infty} \int_c^x F_m(t)\,dt = \int_c^x g(t)\,dt, \text{ uniformly for } c, x \text{ in } (a, b).$$

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