COMPLETELY CONTINUOUS NORMAL OPERATORS WITH PROPERTY L

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1. Introduction. Two matrices A and B are said to have property L if it is possible to arrange their characteristic roots

$$A: \ \lambda_1, \lambda_2, \cdots, \lambda_n$$

$$B: \ \mu_1, \mu_2, \cdots, \mu_n$$

in such a way that for every α, the characteristic roots of $\alpha A + B$ are given by $\alpha \lambda_i + \mu_i$. In [1] this property is investigated, and among other things a conjecture of Kac is confirmed by showing that if A and B are hermitian, then they commute. In [2] this is generalized by replacing "hermitian" by "normal".

In this note we launch the project of generalizing such results to (complex) Hilbert space. However, since it is not clear how to formulate the problem for general operators (especially in the presence of a continuous spectrum), we shall content ourselves with the completely continuous case. For self-adjoint operators we obtain a fully satisfactory generalization (Theorem 1). For the more general case of normal operators we find ourselves obliged to make an extra assumption roughly to the effect that nonzero characteristic roots are paired only to nonzero roots. In the finite-dimensional case such an assumption would be harmless; indeed, by adding suitable constants to A and B, we could even arrange to have all the characteristic roots of A and B nonzero. It would nevertheless be of interest to determine whether this blemish can be removed from Theorem 2.

2. Remarks. Before we state the results, some remarks are in order. The number λ is a characteristic root of A if $A - \lambda I$ has a nonzero null space. If A is a completely continuous normal operator, its characteristic roots are either finite in number or form a sequence approaching zero. We have an orthogonal decomposition of the Hilbert space:

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