EIGENVALUES OF CIRCULANT MATRICES

RICHARD S. VARGA

1. Introduction. The integral equations

$$u(z_j) = \lambda \oint_C A(z, z_j)\, u(z)\, dq + \Phi(z_j), \tag{1}$$

where C is a smooth closed curve, and

$$A(z, z_j) = d \arg (z - z_j)/dq,$$

has many important applications. Thus [6], iteration of (1) gives a solution for the conformal mapping problem for the interior and exterior of C.

In numerical work, the rate of convergence of such iterations depends on the eigenvalues of the integral operator $A(z, z_j)$. It is known that the absolute values of the nontrivial[1] eigenvalues of the integral operator $A(z, z_j)$ are less than one. A recent paper [1] gives a sharper bound to the eigenvalues.

However, in numerical computation, equation (1) must be replaced [6] by a discrete equation of the form

$$u_{r+1}(z_j) = \lambda \sum_{k=1}^{N} A_{jk}\, u_r(z_j) + \Phi(z_j). \tag{2}$$

This makes it important to know the relation between the eigenvalues of $A(z, z_j)$ and those of the matrix A_{jk}.

We determine this relation below in the special case that C is an ellipse. In particular, we show that the eigenvalues of A_{jk} approach $N/2$ times those of $A(z, z_j)$ with exponential convergence. Since trapezoidal integration based on trigonometric interpolation gives exponential accuracy, this fact is probably

[1] It is easy to verify that for the eigenfunction $u(z) \equiv 1$, we have the simple eigenvalue unity. By the nontrivial eigenvalues of $A(z, z_j)$, we mean all other eigenvalues.

Received November 26, 1952. This work was done at Harvard University under Project N5ori-07634 with the Office of Naval Research. The author wishes to express his appreciation to Professor Garrett Birkhoff for helpful suggestions.

Pacific J. Math. 4 (1954), 151-160