ON THE EXISTENCE PROBLEM OF LINEAR PROGRAMMING

JAMES R. JACKSON

1. Notation and introduction. We shall consistently use Latin capital letters to represent real rectangular matrices, lower-case Latin letters for real column vectors, and lower-case Greek letters for real scalars. The appearance of a subscript on a symbol "lowers its classification one step"; thus: A will represent a matrix, A_j will represent the jth column of A (a column vector), A_{ij} will be the entry in the ith row and the jth column of A; b_i will be the ith component of the column vector b; and so on. The dimensions of the matrices and vectors will not always be mentioned, but it is of course implied that they are consistent in the sense that all indicated operations are meaningful; for instance, the appearance of the product Ax implies that the number of columns in the matrix A is equal to the number of components of x. Vector and matrix inequalities are based upon the following notations (where Γ represents either a matrix or a vector):

$$\Gamma = 0 \text{ means each entry in } \Gamma \text{ is zero,}$$

$$\Gamma \geqq 0 \text{ means each entry in } \Gamma \text{ is nonnegative,}$$

$$\Gamma \geq 0 \text{ means } \Gamma \geqq 0, \text{ but } \Gamma = 0 \text{ is false,}$$

$$\Gamma > 0 \text{ means each entry in } \Gamma \text{ is positive.}$$

Several of the proofs given below can be replaced by proofs based on the transposition theorem on linear inequalities [4].

We shall be concerned with the following problem: Given a matrix A and a vector b, does there exist a vector $x \geqq 0$, such that $Ax = b$? Otherwise expressed, we wish to consider the problem of whether the set

$$\{A;\, b\} = \{x \mid Ax = b,\ x \geqq 0\}$$

is nonempty.

In order to eliminate trivial cases, we assume that $b \neq 0$; and also make the obviously nonrestrictive assumption that no column of A is identically zero.

Received February 18, 1953. This paper was prepared under Contract Nonr 233(02), Office of Naval Research.

Pacific J. Math. 4 (1954), 29-36