ON THE GROWTH OF FUNCTIONS HAVING POLES OR ZEROS ON THE POSITIVE REAL AXIS

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1. Introduction. Let Γ denote the boundary of a region Δ containing a right half of the real axis, and $f(z)$ a function holomorphic in Δ except possibly for simple poles on the positive real axis. Through the residue theorem, the integral

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)e^{-zs}\,dz$$

makes correspond to $f(z)$ a Dirichlet series whose coefficients are the residues of $f(z)$ at its poles. It is the purpose of this paper to exhibit some immediate applications, using this familiar device, of the theory of asymptotic Dirichlet series to a study of the growth of functions with poles of bounded order, or zeros of at least a certain order, on the positive real axis. Let the poles, or zeros, occur at points $z = \lambda_n$, where $0 < \lambda_n \uparrow \infty$. Theorems 2A and 2B, in §3, relate the growth of leading coefficients in the Laurent developments about the points $z = \lambda_n$ to the growth of $f(z)$ in Δ. Theorems 3A and 3B, in §4, apply to functions of exponential type in the right half-plane outside Δ, satisfying much weaker growth conditions in Δ. Theorem 3A may be thought of as stating that such a function, with a specified rapidity of decrease on the imaginary axis, is holomorphic if it does not have poles of too great order; while Theorems 3A and 3B together may be regarded as saying that if such a function has "zeros" of minimum order too great for its growth on the imaginary axis, poles counting as negative zeros, then it vanishes identically. (The emphasis on this aspect of Theorem 3A and this interpretation of the uniqueness conclusions of Theorems 3A and 3B were suggested by the referee.)

2. Definitions, the fundamental inequality, and properties of the function $C(z)$. Let $\{\lambda_n\}$ $(n = 1, 2, \cdots)$ be a sequence of positive numbers, strictly increasing without bound $(0 < \lambda_n \uparrow \infty)$, such that

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D^{*} < \infty. \tag{1}$$

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