# CONCERNING TOTAL DIFFERENTIABILITY OF FUNCTIONS OF CLASS $P$

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**1. Introduction.** Since 1900 some eight or more writers have formulated conditions under which a function of two real variables shall be said to be of bounded variation. In two papers Adams and Clarkson [1, 4] examined and compared most of these definitions. In particular, they established the relations

$$\overline{T} \cap M \supset P \supset A, \qquad P \cap M = P,$$

where $\overline{T}$, $P$, and $A$ represent respectively the classes of functions which are of bounded variation in an extended Tonelli sense [1], in the sense of Pierpont, and in the sense of Arzelà [4], and $M$ stands for the class of plane measurable functions. An explicit definition of the class $P$ will be given presently. For other definitions see Clarkson and Adams [4].

Burkill and Haslam-Jones [2] have shown that each function in $A$ is totally differentiable almost everywhere. Adams and Clarkson [1] have proved that each function in $\overline{T} \cap M$ is approximately totally differentiable almost everywhere, although not necessarily totally differentiable anywhere. The question of whether each function in $P$ is totally differentiable almost everywhere has been left open; the object of the present paper is to settle this question.

Saks [6] has shown that in a certain subset $E \subset \overline{T} \cap M$, suitably metrized, the functions which are nowhere totally differentiable form a residual set. One might naturally raise the question as to the category of the set $P \cap E$, for if this set were of second category in $E$, our question would be answered at once; but it turns out that $P \cap E$ is of first category in $E$.

In this paper (see §§3, 4, and 5) we show by exhibiting an example constructed along lines suggested by A. P. Morse that there exist functions which are in $P$ and which are nowhere totally differentiable. We then show (see §§6-10 and §11) that functions which are nowhere totally differentiable form residual sets (complements of sets of first category) in the classes $P \cap C$ and $P \cap E$, where $C$ represents the class of functions continuous on the unit

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