ASYMPTOTIC LOWER BOUNDS FOR THE FREQUENCIES OF CERTAIN POLYGONAL MEMBRANES

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1. Background. Let the bounded, simply connected, open region R of the (x, y) plane have the boundary curve C. If a uniform elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, the square $\lambda = \lambda(R)$ of the fundamental frequency satisfies the conditions (subscripts denote differentiation)

$$(1a) \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u \quad \text{in } R, \\ \lambda = \text{minimum}, \end{cases}$$

with the boundary condition

$$(1b) \qquad u(x, y) = 0 \quad \text{on } C.$$

The solution u of problem (1) is unique up to a constant factor. It is known [**13**, p. 24] that λ is the minimum over all piecewise smooth functions u satisfying (1b) of the Rayleigh quotient

$$(2) \qquad \rho(u) = \iint_R |\nabla u|^2 dxdy \Big/ \iint_R u^2 dxdy,$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. In many practical methods for approximating λ one essentially determines $\rho(u)$ for functions u satisfying (1b) which are close to a solution of the boundary value problem (1). See [**9**, p. 112; **6**, p. 276; **11**, and **12**]. By (2) these approximations are known to be *upper bounds* for λ; they can be made arbitrarily good with sufficient labor. It is obviously of equal importance to obtain close lower bounds for λ; cf. [**14**].

The lower bounds for λ given by Pólya and Szegö [**13**] are ordinarily far

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