COMMUTING SPECTRAL MEASURES ON HILBERT SPACE

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1. Introduction. By a "spectral measure" on Hilbert space H we mean a family of bounded operators $E(\sigma)$ on H defined for all Borel sets σ in the plane. We suppose:

(i) If σ_0 denotes the empty set and σ_1 the whole plane, then

$$E(\sigma_0) = 0, \quad E(\sigma_1) = I,$$

where I is the identity.

(ii) For all σ_1, σ_2,

$$E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2);$$

and for disjoint σ_1, σ_2,

$$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2).$$

(iii) There exists a constant M with $||E(\sigma)|| \leq M$, all σ. It follows that $E(\sigma)^2 = E(\sigma)$ for each σ, and $E(\sigma_1)E(\sigma_2) = 0$ if σ_1, σ_2 are disjoint.

Mackey has shown in [3], as part of the proof of Theorem 55 of [3], that if $E(\sigma)$ is a spectral measure with the properties just stated, then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ is self-adjoint for every σ. In a special case this result was proved by Lorch in [2]. We shall prove:

THEOREM 1. *Let $E(\sigma)$ and $F(\eta)$ be two commuting spectral measures on H; that is,*

$$E(\sigma)F(\eta) = F(\eta)E(\sigma)$$

for every σ, η. Then there exists a bicontinuous operator A such that $A^{-1}E(\sigma)A$ and $A^{-1}F(\eta)A$ are self-adjoint for every σ, η.

As a corollary of Theorem 1, we shall obtain:

Received March 4, 1953.

Pacific J. Math. 4 (1954), 355-361