ON THE DIMENSION THEORY OF RINGS (II)

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1. Introduction. As in [3], we shall say that an integral domain O is n-dimensional if in O there is a proper chain

$$(0) \subset P_1 \subset \cdots \subset P_n \subset (1)$$

of prime ideals, but no such chain

$$(0) \subset P_1' \subset \cdots \subset P_{n+1}' \subset (1).$$

In Theorem 2 of [3] it was shown that if O is n-dimensional, then $O[x]$ is at least $(n+1)$-dimensional and at most $(2n+1)$-dimensional: here, as throughout, x is an indeterminate. After preparatory constructions in Theorems 1 and 2 below, this theorem is completed in Theorem 3 by showing that for any integers m and n with $n+1 \leq m \leq 2n+1$, there exist n-dimensional rings O such that $O[x]$ is m-dimensional. The other theorems mainly concern 1-dimensional rings. Such rings O can be divided into those for which $O[x]$ is 2-dimensional and those for which this condition fails, the so-called F-rings. The paper [3] was concerned with the existence of F-rings and showed [3, Theorem 8] that the 1-dimensional ring O is not an F-ring if and only if every quotient ring of the integral closure of O is a valuation ring. Below, in Theorem 5, we show more generally that if O is 1-dimensional but not an F-ring, then $O[x_1, \cdots, x_n]$ is $(n+1)$-dimensional, where the x_i are indeterminates: this theorem depends on the essentially more general Theorem 4, which says that if O is an m-dimensional multiplication-ring, then $O[x_1, \cdots, x_n]$ is $(m+n)$-dimensional. In the case that the x_i are not indeterminates, one can still say (Theorem 10) that

$$\dim O[x_1, \cdots, x_n] = 1 + \text{degree of transcendency of } O[x_1, \cdots, x_n]/O,$$

provided that the intersection of the prime ideals $(\neq (0))$ in O is $= (0)$, where O is a 1-dimensional ring such that $O[x]$ is 2-dimensional. For F-rings O, Theorem 6 shows that

$$n+2 \leq \dim O[x_1, \cdots, x_n] \leq 2n+1,$$

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